Permutations and Combinations

Finite Math

23 April 2019

Poker Hands!

Example

Suppose we ave a standard 52-card deck and we are considering 5-card poker hands.

(a) *How many hands have 3 hearts and 2 spades?*

(b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*

(c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*

(d) *How many possible 3 of a kinds are there? (The other two cards have a different number from the 3 of a kind and from each other.)*

(e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*

What is Probability?

If an experiment is repeated several times (such as rolling a die 100 times or flipping a coin 100 times), there is no guarantee, and in fact it is highly unlikely, that the same result will appear every time. This type of experiment is a *random experiment*. Probability is a field of mathematics dedicated to studying experiments with random outcomes.

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There are some language differences when we are focused on probability in the real world versus theoretically:

Real World	Theoretical
Experiment	Sample space
Outcome	Event

In class, we will likely not make a distinction between the two, and probably even use them interchangeably.

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Consider the number wheel with numbers 1-18 *on it.*

(a) *What is the sample space?*

(b) *What is the event corresponding to the outcome of the experiment being prime? Is it a simple event or a compound event?*

(c) *What is the event corresponding to the outcome of the experiment being the square of* 4*? Is it a simple event or a compound event?*

Now You Try It!

Example

Consider again the number wheel with numbers 1-15 *on it.*

(a) *What is the sample space?*

(b) *What is the event corresponding to the outcome of the experiment being divisible by* 12*? Is it a simple event or a compound event?*

(c) *What is the event corresponding to the outcome of the experiment being even? Is it a simple event or a compound event?*

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An experiment consists of recording the boy-girl composition of a three-child family. What would be an appropriate sample space if:

(a) *we are interested in the genders of the children in the order of their births? (A tree diagram can help.)*

(b) *we are interested only in the number of girls in family?*

(c) *we are interested only in which gender there are more of?*

(d) *we are interested in all three items from (a)-(c)?*

Now You Try It!

Solution

(a)

$$S_1 = \{GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB\}$$

(b)

$$S_2 = \{0, 1, 2, 3\}$$

(c)

$$S_3 = \{G, B\}$$

(d) *Use* S_1.

Dice Rolling Outcomes

Example

Suppose we run an experiment by rolling two dice. What is the most fundamental sample space for this experiment? Give the event for each of the following outcomes. Which are simple events?

(a) *A sum of* 7 *turns up.*

(b) *A sum of* 11 *turns up.*

(c) *A sum less than* 4 *turns up.*

(d) *A sum of* 12 *turns up.*

(e) *A sum of* 5 *turns up.*

(f) *A sum which is prime turns up.*

(g) *A sum of* 2 *turns up.*

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Definition (Probabilities of Simple Events)

Given a sample space

$$S = \{e_1, e_2, ..., e_n\}$$

with n simple events, to each simple event e_i we assign a real number, denoted by $P(e_i)$, called the probability of event e_i. These numbers can be assigned in an arbitrary manner as long as the following two conditions are satisfied:

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Any probability assignment that satisfies Conditions 1 and 2 is said to be an acceptable probability assignment.

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Notice that this satisfies the two conditions of the previous definition. Is this a reasonable assignment of probabilities? Ostensibly yes since there are only two outcomes of a coin flip and there is no reason to doubt that the coin is *fair*, i.e., the two outcomes are equally likely.

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Another assignment we could technically make is

$$P(H) = 1 \qquad P(T) = 0.$$

While this does fit the rules of an acceptable probability assignment, it is not *reasonable* in this case, unless the coin had two heads.

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since $P(H) + P(T) = 1.4 > 1$.

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(c) *If E is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in E.*

(d) *If E is the sample space S, then $P(E) = P(S) = 1$ (this is a special case of part (c).)*

Example

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Refer back to the example where we roll two dice. If we assume that every simple event is equally likely:

(a) *What is the probability of a simple event happening?*

(b) *What are the possible numbers that the two dice could add up to?*

(c) *What are the probability of each of the events in part (b) happening?*

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$$P(H) = \frac{373}{1000} \qquad P(T) = \frac{623}{1000}$$

since it reflects the results of an extensive experiment.

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Equally Likely Assumption

When we were talking about assigning probabilities of 0.5 to heads and 0.5 to tails for flipping a coin, and a probability of $\frac{1}{6}$ for any number to come up when rolling a 6-sided die, we are making an assumption on the probabilities of the experiment called an *equally likely assumption*.

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Generally, if the sample space is

$$S = \{e_1, e_2, ..., e_n\},$$

we assign to each e_i a probability of $\frac{1}{n}$ since there are n possible outcomes and we want each of them to be equally likely. This gives us the following theorem...

Equally Likely Assumption

Theorem (Probability of an Arbitrary Event under an Equally Likely Assumption)

If we assume that each simple event in a sample space S is equally likely to occur, then the probability of an arbitrary event E in S is given by

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We saw this theorem in action when we found the theoretical probabilities for rolling a number on a pair of dice.